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Classical Spin in a Potential Field

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We consider an ensemble of restricted discrete random walks in 2 + 1 dimensions. The restriction on the walks is such as to give particles an intrinsic angular momentum. The walks are embedded in a field which affects the mean free path of the walks. We show that the dynamics of the walks is such that second-order effects are described by a discrete form of Schrödinger's equation for particles in a potential field. This provides a classical context of the equation which is independent of its quantum context.

1. INTRODUCTION

In nonrelativistic quantum mechanics the dynamics of wave functions is governed by Schrödinger's equation. However, in the theory itself the wave function is simply a mathematical device which helps in the calculation of *observables*. The existence (or lack thereof) of a physical analog of wave functions is not something which is verifiable from the theory itself, since wave functions are not themselves observable. Furthermore, although wave functions propagate as waves, one can in reality always choose to observe particles. This peculiar feature of quantum mechanics is not an intrinsic feature of Schrödinger's equation. It is a result of the interpretation of the equation in the context of quantum mechanics. Part of the objective of this article is to confirm that Schrödinger's equation can appear in a context in which the wave function has a physical counterpart, and is not just a formal device for the calculation of observables.

In order to construct a context in which the solutions of Schrödinger's equation describe features of real physical systems we shall consider a random-walk model of diffusion. There has always been an interest in the relation between diffusion theory, which has a microscopic model in Brownian motion,

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and Quantum Mechanics, which is currently without a satisfactory microscopic model. In the 1940s Feynman (1948) invented his path integral approach to quantum mechanics in which the wave function became a sum over Brownian-like paths of a complex exponential. This sum over paths is related to the classical Wiener integral of diffusion by a formal analytic continuation (Feynman and Hibbs, 1965; Schulman, 1981). Subsequently contact has been made between the Schrödinger and diffusion equations by proposing reversible diffusion processes (Nelson, 1985; Nagasawa, 1993; Nottale, 1993; El Naschie, 1995). An alternative to these approaches is provided by the de Broglie–Bohm interpretation of quantum mechanics (Holland, 1993) and a collection of current work on understanding the relation between quantum and classical physics is provided in El Naschie *et al.* (1995, 1996a, b) and El Naschie and Prigogine (1996).

Since we cannot observe wave functions directly in their quantum context, it seems reasonable to look for contexts for Schrödinger's equation where the solutions have a physical counterpart. There are now several such systems available, and in these models the objects of study are ensembles of classical particles where the systems exhibit quantum dynamics as second-order effects. The nonrelativistic free particle in $1 + 1$ dimensions is considered in (Ord, 1996a) and Ord and Deakin (1996), and a 16-state $(2 + 1)$-dimensional model may be found in Ord and Deakin (1997). Relativistic versions may be found in Ord (1996b, c).

In this work we show that a simple model of a diffusing particle with intrinsic angular momentum (Ord, 1993) produces Schrödinger's equation for a particle in a potential field in $2 + 1$ dimensions as a second-order effect. The model uses only a four-state history and is thus simpler than a previous 16-state version (Ord and Deakin, 1997).

2. CALCULATION

Assume that particles hop along the principal diagonals where the adjacent nodes on the space-time lattice form a rectangular box whose dimensions in space and time are δ and ϵ, respectively. If the space axes are labeled x and y, then at each time step the particle moves a distance $\pm\delta$ along both axes and a distance ϵ along the time axis. The model we consider has the unusual feature that at each space-time node on the lattice, the particle can either maintain its direction or it can make a *turn to the left*. That is, if we look at the projection of a path in the xy plane, this projection consists of directed line segments that are perpendicular and spiral to the left. We label the particle in state 1 if the projections of the path on the x axis and the y axis both have positive components. The remaining three states are defined

Table I

State μ	Projection on x axis	Projection on y axis
1	Positive	Positive
2	Negative	Positive
3	Negative	Negative
4	Positive	Negative

as illustrated in Table I. Suppose a particle starts in state 1. As it moves along the lattice, it will change to state 2, followed by state 3, and then state 4, before returning again to state 1. Since the particles spiral to the left, there is an intrinsic angular momentum associated with each particle.

Let $p_\mu(m\delta, n\delta, s\epsilon)\delta^2$ be the probability that a particle leaves $(m\delta, n\delta)$ at time $s\epsilon$ in state μ ($\mu = 1, \ldots, 4$). The difference equations are

$$\begin{aligned}
&p_1(m\delta, n\delta, (s+1)\epsilon)\\
&\quad = \alpha p_1((m-1)\delta, (n-1)\delta, s\epsilon) + \beta p_4((m-1)\delta, (n+1)\delta, s\epsilon)\\
&p_2(m\delta, n\delta, (s+1)\epsilon)\\
&\quad = \beta p_1((m-1)\delta, (n-1)\delta, s\epsilon) + \alpha p_2((m+1)\delta, (n-1)\delta, s\epsilon)\\
&p_3(m\delta, n\delta, (s+1)\epsilon)\\
&\quad = \beta p_2((m+1)\delta, (n-1)\delta, s\epsilon) + \alpha p_3((m+1)\delta, (n+1)\delta, s\epsilon)\\
&p_4(m\delta, n\delta, (s+1)\epsilon)\\
&\quad = \beta p_3((m+1)\delta, (n+1)\delta, s\epsilon) + \alpha p_4((m-1)\delta, (n+1)\delta, s\epsilon)
\end{aligned} \tag{1}$$

where $\alpha + \beta = 1$. Here α is the probability that a particle maintains its direction at the next time step, and β is the probability that a particle will change its direction at the next time step. The governing equations (1) have a straightforward interpretation. The first equation in (1) implies that the probability $p_1\delta^2$ that the particle leaves the node $(m\delta, n\delta, (s+1)\epsilon)$ in state 1 is equal to the sum of two probabilities:

- $\alpha p_1\delta^2$, the probability that the particle leaves the node $((m-1)\delta, (n-1)\delta, s\epsilon)$ in state 1 and remains in this state when it leaves $(m\delta, n\delta, (s+1)\epsilon)$.
- $\beta p_4\delta^2$, the probability that the particle leaves $((m-1)\delta, (n+1)\delta, s\epsilon)$ in state 4 and changes to state 1 when it leaves the node $(m\delta, n\delta, (s+1)\epsilon)$.

We impose the condition

$$\sum_{\mu=1}^{4} \sum_{m,n=-\infty}^{+\infty} p_\mu(m\delta, n\delta, s\epsilon)\delta^2 = 1 \tag{2}$$

which establishes the fact that the probability that a particle is somewhere on the lattice at a given time is one. Once the initial conditions are given, (1) has a unique solution.

The parameters ϵ and δ are related by the requirement that in the diffusive continuum limit, $\delta/(2\epsilon) \rightarrow D$ as $\delta \rightarrow 0$, where D is the diffusion constant. Hence, we have, for small δ,

$$\frac{\delta^2}{2\epsilon} = D + O(\delta) \qquad \text{or} \qquad \epsilon = \frac{\delta^2}{2D} + O(\delta^3) \tag{3}$$

We put in a potential field through α. Here we imagine that the lattice walkers choose their next state according to a canonical ensemble in which a smooth bounded potential $v(x)\epsilon$ acts like an energy. That is, suppose

$$\alpha = \frac{e^{-v(x)\epsilon}}{e^{-v(x)\epsilon} + e^{v(x)\epsilon}} \tag{4}$$

so that

$$\alpha = \frac{1}{2}(1 - v(x)\epsilon) + O(\epsilon^2) = \frac{1}{2}\left(1 - \frac{v(x)}{2D}\delta^2\right) + O(\delta^3) \tag{5}$$

We define

$$p = [p_1, p_2, p_3, p_4]^T$$

$$E_x^{\pm 1} p(m\delta, n\delta, s\epsilon) = p(m\delta \pm \delta, n\delta, s\epsilon)$$

$$E_y^{\pm 1} p(m\delta, n\delta, s\epsilon) = p(m\delta, n\delta \pm \delta, s\epsilon)$$

$$E_t p(m\delta, n\delta, s\epsilon) = p(m\delta, n\delta, s\epsilon + \epsilon)$$

We have

$$E_t p(m\delta, n\delta, s\epsilon) = Tp(m\delta, n\delta, s\epsilon) \tag{6}$$

$$T = \begin{bmatrix} \alpha E_y^{-1}E_x^{-1} & 0 & 0 & \beta E_y E_x^{-1} \\ \beta E_y^{-1}E_x^{-1} & \alpha E_y^{-1}E_x & 0 & 0 \\ 0 & \beta E_y^{-1}E_x & \alpha E_y E_x & 0 \\ 0 & 0 & \beta E_y E_x & \alpha E_y E_x^{-1} \end{bmatrix} \tag{7}$$

The equation we consider is

$$E_t^k p(M\delta, N\delta, S\epsilon) = (T)^k p(M\delta, N\delta, S\epsilon) \tag{8}$$

which follows from (6) by applying the operator E_t repeatedly. This equation expresses p at $t = (S + k)\epsilon$ in terms of p at $t = S\epsilon$. Suppose we are interested in (x, y, t) in a neighborhood of a fixed point (X, Y, T) in space-time. Given δ and ϵ, we select the node $(m\delta, n\delta, s\epsilon)$ as $(M\delta, N\delta, S\epsilon)$ such that

$$\begin{aligned} &M\delta \le X < (M + 1)\delta, \quad N\delta \le Y < (N + 1)\delta \\ &S\epsilon \le T < (S + 8)\epsilon, \quad S = 0 \bmod 8 \end{aligned} \tag{9}$$

We will show that the solution of (6) cannot be approximated by a continuous function for arbitrarily small δ; however, we can approximate $p(m\delta, n\delta, s\epsilon)$ provided we restrict time to $t = s\epsilon$ ($s = 0, 8, 16, \ldots$) in (8). Then the solution is approximated by a continuous function, which is a sum of solutions of partial differential equations when $k = 8l$ ($l = 1, 2, \ldots$) in (8).

To approximate the solutions for small δ, we expand the shift operators as power series in δ as follows:

$$\begin{aligned} E_x^{\pm 1} &= 1 \pm \delta\frac{\partial}{\partial x} + \frac{1}{2}\delta^2\frac{\partial^2}{\partial x^2} + O(\delta^3) \\ E_y^{\pm 1} &= 1 \pm \delta\frac{\partial}{\partial y} + \frac{1}{2}\delta^2\frac{\partial^2}{\partial y^2} + O(\delta^3) \end{aligned} \tag{10}$$

$$E_t = 1 + \epsilon\frac{\partial}{\partial t} + O(\epsilon^2) = 1 + \frac{\delta^2}{2D}\frac{\partial}{\partial t} + O(\delta^3) \tag{11}$$

We determine the eigenvectors and eigenvalues of the operator T, $(T - \lambda_j I)|v_j\rangle = 0$, where we use bracket notation to represent the vectors, i.e., $|v_j\rangle = v_j$ are column vectors and $\langle v_j| = v_j^\dagger$, which is the complex conjugate transpose of $|v_j\rangle$. We expand T, λ_j, and $|v_j\rangle$ in a power series in δ: $T \sim T^0 + T^1\delta + T^2\frac{1}{2}\delta^2$, $\lambda_j \sim \lambda_j^0\delta + \lambda_j^1\delta + \lambda_j^2\frac{1}{2}\delta^2$, and $|v_j\rangle \sim |v_j^0\rangle + |v_j^1\rangle\delta + |v_j^2\rangle\frac{1}{2}\delta^2$. Substituting these power series into (6), we have the following set of equations to solve:

$$(T^0 - \lambda_j^0 I)|v_j^0\rangle = 0 \tag{12}$$

$$(T^0 - \lambda_j^0 I)|v_j^1\rangle = -T^1|v_j^0\rangle + \lambda_j^1|v_j^0\rangle \tag{13}$$

$$(T^0 - \lambda_j^0 I)|v_j^2\rangle = -T^2|v_j^0\rangle - 2T^1|v_j^1\rangle + \lambda_j^2|v_j^0\rangle + 2\lambda_j^1|v_j^1\rangle \tag{14}$$

The matrices T^i are expressed in terms of the matrix $M(i)$,

$$M(i) = \frac{1}{2}\begin{bmatrix} (t_{11})^i & 0 & 0 & (t_{10})^i \\ (t_{11})^i & (t_{01})^i & 0 & 0 \\ 0 & (t_{01})^i & (t_{00})^i & 0 \\ 0 & 0 & (t_{00})^i & (t_{10})^i \end{bmatrix} \tag{15}$$

as follows: $T^0 = M(0)$, $T^1 = M(1)$, $T^2 = M(2) + (T^0 - I)v(x)/D$, $t_{ij} = (-1)^i\partial/\partial x + (-1)^j\partial/\partial y$.

The eigenvalues and the corresponding eigenvectors for T^0 are readily determined. We have $T^0|v_j^0\rangle = \lambda_j^0|v_j^0\rangle$, where

$$\lambda_1^0 = 0, \qquad \lambda_2^0 = (1 - i)/2, \qquad \lambda_3^0 = (1 + i)/2, \qquad \lambda_4^0 = 1 \tag{16}$$

and the normalized eigenvectors ($\langle v_k^0|v_j^0\rangle = \delta_{kj}$) are

$$|v_1^0\rangle = \frac{1}{2}(-1, 1, -1, 1)^T \qquad |v_2^0\rangle = \frac{1}{2}(i, -1, -i, 1)^T$$

$$|v_3^0\rangle = \frac{1}{2}(-i, -1, i, 1)^T \qquad |v_4^0\rangle = \frac{1}{2}(1, 1, 1, 1)^T$$

These eigenvectors have the property that $\langle v_j^0|(T^0 - \lambda_j^0 I) = 0$.

To determine the solution of (13), we multiply the equation on the left by $\langle v_k^0|$ to obtain

$$\langle v_k^0|T^1|v_j^0\rangle = \lambda_j^1\delta_{kj} + (\lambda_j^0 - \lambda_k^0)\langle v_k^0|v_j^1\rangle \tag{17}$$

For $k = j$, we have $\lambda_j^1 = \langle v_j^0|T^1|v_j^0\rangle$ and we can readily show that $\lambda_j^1 = 0$. For $k \neq j$, $\langle v_k^0|v_j^1\rangle = \langle v_k^0|T^1|v_j^0\rangle/(\lambda_j^0 - \lambda_k^0)$ and, upon defining the right-hand side of this equation to be τ_{kj}, the general solution of (13) is $|v_j^1\rangle = \Sigma_k \tau_{kj}|v_k^0\rangle$, where τ_{jj} is arbitrary. These coefficients have the following values: $\tau_{1j} = 0$ ($j \neq 1$), $\tau_{41} = \tau_{32} = \tau_{23} = 0$, $\tau_{21} = \tau_{31}^* = (\partial/\partial x + i\partial/\partial y)(1 + i)/2$, $\tau_{24} = \tau_{34}^* = (\partial/\partial x - i\partial/\partial y)(1 + i)/2$, and $\tau_{42} = \tau_{43}^* = \partial/\partial x + i\partial/\partial y$.

Similarly, to determine the solution of (14), we multiply the equation on the left by $\langle v_k^0|$ to obtain

$$\langle v_k^0|T^2|v_j^0\rangle + 2\langle v_k^0|T^1|v_j^1\rangle = \lambda_j^2\delta_{jk} + (\lambda_j^0 - \lambda_k^0)\langle v_k^0|v_j^2\rangle \tag{18}$$

For $k \neq j$, we solve this equation for $\langle v_k^0|v_j^2\rangle \equiv \sigma_{kj}$, so that the general solution of (14) is $|v_j^2\rangle = \Sigma_k \sigma_{kj}|v_k^0\rangle$, where σ_{jj} is arbitrary. For $k = j$, we have an expression for λ_j^2 that can readily be expressed in the form

$$\lambda_j^2 = \langle v_j^0|M(2)|v_j^0\rangle + (\lambda_j^0 - 1)v(x)/D + 2\sum_k \tau_{jk}\tau_{kj}(\lambda_k^0 - \lambda_j^0) \tag{19}$$

so that $\lambda_1^2 = -v(x)/D$, $\lambda_2^2 = (\lambda_3^2)^* = \frac{1}{2}(1 + i)\,(\nabla^2 - v(x)/D)$, and $\lambda_4^2 = \nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Thus, we have $T = V \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)V^{-1}$, where $V = [v_1, v_2, v_3, v_4]$. With a change of variables $p = V\tilde{z}$ in (8), we obtain $E_t^k\tilde{z} = \operatorname{diag}((\lambda_1)^k, \ldots, (\lambda_4)^k)\tilde{z}$. The variables are scaled with a change of variables from $\tilde{z}$ to $z = (z_1, z_2, z_3, z_4)^T$, where we choose the normalization appropriate to the continuum limit; that is, $z_i = \tilde{z}_i$ $(i = 1, 4)$, $z_i = 2^{s/2}\tilde{z}_i$ $(i = 2, 3)$. We obtain

$$E_t^k z = \operatorname{diag}((\lambda_1)^k, (\sqrt{2}\lambda_2)^k, (\sqrt{2}\lambda_3)^k, (\lambda_4)^k)z \qquad (20)$$

To approximate z by a continuous function, we expand E_t^k using (11). Consequently, the $O(1)$ terms cancel only if k is a muliple of 8. Setting $k = 8$, and expanding E_t^8 from (11), we can readily show that $z_1(M\delta, N\delta, S\epsilon) = 0$ to all orders of δ. Taking the initial condition $z_1 = 0$, then z_1 is identically zero. For the remaining variables, we define $z_i(M\delta, N\delta, S\epsilon) = z_i^0(M\delta, N\delta, S\epsilon) + O(\delta)$. Substituting the expressions for λ_j^2 into (20) and equating terms of $O(\delta^2)$, we find that $z_i^0(x, y, t)$ is the solution of the partial differential equations

$$\frac{\partial}{\partial t} z_4^0 = D\nabla^2 z_4^0$$

$$i\frac{\partial}{\partial t} z_2^0 = (-D\nabla^2 + v(x))z_2^0, \qquad -i\frac{\partial}{\partial t} z_3^0 = (-D\nabla^2 + v(x))z_3^0 \qquad (21)$$

Thus, z_4^0 is the solution of the diffusion equation. This result is expected on physical grounds because the potential only affects the local mean free path and does not favor either direction. Since the mean free path is zero in the continuum limit, z_4^0 does not contain the finite potential in this limit. For the other variables, z_2^0 is a solution of Schrödinger's equation and z_3^0 is a solution of the conjugate Schrödinger equation.

In conclusion, on the lattice, p must be real, so that we must take $z_3 = z_2^*$. Thus, p in terms of z is

$$p = v_4 z_4 + \frac{1}{2^{s/2}}(v_2 z_2 + v_2^* z_2^*) \qquad (s = 0 \bmod 8) \qquad (22)$$

3. SUMMARY

In the above calculation we discovered Schrödinger dynamics in the description of ensembles of classical particles. It is worth noting that there is no analytic continuation involved in this result. The complex nature of the wave function solutions of the Schrödinger equations in (21) arise because of the actual behavior of ensembles of particles. We do not have to invoke a formal analogy or introduce time-symmetric diffusion. The interference effects implicit in the solutions arise from the fact that forcing the particles to have a net angular momentum and then subtracting contributions from opposite directions is like building a vacuum of particles and antiparticles. The microscopic angular momentum ensures that the *virtual particles* cannot erase themselves through symmetry. [If one allows particles to go left or right with equal probability, than the λ_j^0 of equation (16) are all real and there is no Schrödinger dynamics.] Note, however, that the wavelike solutions of the resulting Schrödinger equation arise not from an interacting fluid of particles, but as a pattern forming in an ensemble of *noninteracting* particles. Thus the solutions represent ensemble averages of single-particle features.

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